Games on Temporal Graphs

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Temporal graphs are graphs where the edge relation changes over time. They are often presented as a sequence $G_0, G_1, \ldots$ of graphs over the same set of vertices. Equivalently, one can define them as pairs $G = (V, E)$ consisting of a set V of vertices and associated edge availability predicate $E : V^2 \to 2^{\mathbb{N}}$ that determines at which integral times a directed edge can be traversed. This model has been used to analyse dynamic networks and distributed systems in dynamic topologies, such as gossiping and information dissemination [15]. There is also a large body of work that considers temporal generalisations of various graph-theoretic notions and related algorithmic questions. We refer to [9,13] for overviews of temporal graph theory and its applications.

We note that most works on temporal graphs assume a finite horizon, and are presented as a sequence of graphs G_0, G_1, $\ldots G_k$ up to the horizon k. Some works on periodic temporal graphs [8,6] assume that the edge availability relation repeats periodically with time. As a contribution towards for better modelling, we consider temporal graphs where the edge availability relation is given by an existential Presburger formula $\phi_e(x)$, such that the edge e is available at time t if and only if $\phi_e(t)$ is true. This allows succinct encoding of temporal graphs, while also capturing (ultimately) periodic temporal graphs. Note that while we introduce this encoding, most of our results do not rely on this specific encoding of the edge availability relation.

In this talk, we will consider the complexity of solving games played on (succinct) temporal graphs. Two player zero-sum verification games on directed graphs play a central role in reactive synthesis [14]. Here, a controllable system and an antagonistic environment are modeled as a game in which two opposing players jointly move a token through a graph. States are either owned by Player 1 (the system) or Player 2 (the environment), and the owner of the current state picks a valid successor. Such a play is won by Player 1 if, and only if, the constructed path satisfies a predetermined *winning condition* that models the desired correctness specification. The core algorithmic problem is solving games: to determine which player has a strategy to force a win, and if so, how.

Determining the complexity of solving games on static graphs, i.e., graphs where edge availability does not change with time, has a long history and continues to be an active area of research [7]. We recall here only that solving reachability games, where Player 1 aims to eventually reach a designated target state, is complete for polynomial time. The precise complexity of solving parity games is a long-standing open question. It is known to be in UP ∩ coUP [11], and so in particular in NP and coNP, and recent advances have led to quasi-polynomial time algorithms [4].

Games on temporal graphs with a finite horizon, or period of some absolute value k given in binary can be seen as games on exponentially succinctly presented arenas. Unfolding them up to time k yields an ordinary game on the exponential sized graph which allows to transfer upper bounds, that are not necessarily optimal.

Parity games on temporal graphs are closely related to timed-parity games, which are played on the configuration graphs of timed automata [1]. However, the time in temporal graphs is discrete as opposed to the continuous time semantics in timed automata. Solving timed parity games is complete for EXPTIME [12,5] and the lower bound already holds for reachability games on timed automata with only two clocks [10]. Unfortunately, a direct translation of (games on) temporal graphs to equivalent timed automata games requires at least two clocks: one to hold the global time used to check the edge predicate and one to ensure that time progresses one unit per step.

We studied the complexity of solving parity games on temporal graphs. As a central variant of independent interest are what we call *punctual* reachability games, that are played on a static graph and player wants to reach a target vertex at a given binary encoded time. We show that solving such games is already hard for PSPACE, which provides a lower bound for all temporal graph games we consider.

We show how to solve parity games on (ultimately) periodic temporal graphs. The difficulty to overcome here is that the period may be exponential in the number of vertices and thus a naïvely solving the game on the unfolding only yields algorithms in exponential space. Our approach relies on the existence of polynomially sized summaries that can be verified in PSPACE using punctual reachability games.

We also consider the complexity of solving games on temporal graphs with explorability objective, which requires the explorer player to visit all vertices of the graph. Explorability is a well studied problem on temporal graphs. On static graphs, explorability games can be solved in polynomial time. However, on explicitly presented temporal graphs (as a sequence of graphs on same set of vertices), we show that determining the winner of explorability game is PSPACE-complete. We leave the exact complexity of solving explorability games on succinctly presented temporal graphs open, in between PSPACE and EXPTIME.

Assuming that the edge availability relation is given in existential fragment of the Presburger arithmetic, we also show that deciding reachability in (1-player) temporal graph is in fact PSPACE-complete and therefore the same complexity as (2-player) reachability games on (succinctly represented) temporal graphs.

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